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Solutions of the Hyperbolic sine–Gordon Equations

C. Hoenselaers[1]

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We study soliton solutions of the Sinh–Gordon equation. It is also shown that the Cosh–Gordon equation, whilst being integrable, does not admit pure solitons.

1. INTRODUCTION

The sine–Gordon equation

$$\left(\partial_x^2 - \partial_t^2\right)\varphi = \partial_{uv}\varphi = \sin\varphi$$

(in space-time or double null coordinates) has been studied in many publications too numerous to list here, as one instance cf. Rogers and Schief (2002). Also, replacing the sine-function by the cosine-function does not yield anything new.

On the other hand, replacing the sine-function with its hyperbolic counterpart might lead to some new insights. In this short note we shall examine the hyperbolic counterparts – there are two of them, since there is no real transformation between the Sinh and the Cosh – of the sine–Gordon equation.

2. GENERAL REMARKS

We study the equations

$$\partial_{uv}\varphi = \mathrm{Sinh}\,\varphi \tag{1a}$$

$$\partial_{uv}\varphi = \mathrm{Cosh}\,\varphi \tag{1b}$$

Both equations are known to be integrable and admit a pseudopotential, viz.

$$\partial_u\gamma = \partial_u\varphi - \frac{1}{\lambda}\mathrm{Sinh}\,\gamma$$

[1] Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3HE, UK
e-mail: C.A.Hoenselaers@lboro.ac.uk

$$\partial_v \gamma = \lambda \text{ Sinh}(\varphi - \gamma) \tag{2a}$$

$$\partial_u \gamma = \partial_u \varphi + \frac{1}{\lambda} \text{Sinh}\, \gamma$$
$$\partial_v \gamma = \lambda \text{ Cosh}(\varphi - \gamma) \tag{2b}$$

The integrability conditions for (2a) and (2b) yield the Eqs. (la) and (1b). With the help of the pseudopotential a new solution is given in both cases via a Bäcklund transformation as

$$\tilde{\varphi} = \varphi - 2\gamma \tag{3}$$

Each Bäcklund transformation generates a soliton. The difference between the two equations is that for the Sinh–Gordon equation one has a trivial solution, $\varphi = 0$, to start the process whereas for the Cosh–Gordon equation this solution does not exist. This leads to the suspicion that the Cosh–Gordon equation, whilst being integrable, does not admit pure solitons.

Since there appears to be no generally accepted definition of a "soliton," we shall denote, in the present context, a localized "lump" of energy, to be described by the Hamiltonian of the system in question.

3. SOME SOLUTIONS

We shall first present solutions for the Sinh–Gordon equation in space-time coordinates, viz.

$$\partial_x^2 \varphi - \partial_t^2 \varphi = \text{Sinh}\, \varphi \tag{4}$$

We take this solution to be a travelling wave which, by the Lorentz invariance of the equation, can be taken to be in its rest frame, hence $\varphi = \varphi(\mathrm{x})$. Multiplying (4) by $\partial_x \varphi$ and integrating we get

$$\frac{1}{2} \partial_x \varphi^2 = \text{Cosh}\, \varphi - E \tag{5}$$

the solution of which is given in terms of Jacobi elliptic functions (in the notation we follow Gradshteyn and Ryzhik, 1965) as

$$\varphi = 2 \ln \left(\sqrt{k}\, sn \left(\frac{x}{2\sqrt{k}}, k \right) \right), \quad E = \frac{1}{2} \left(k + \frac{1}{k} \right) \tag{6}$$

In particular, for $k = 1$, $E = 1$, this solution degenerates into the 1-soliton solution which can be rewritten as

$$\varphi = 4\ Ar \text{ Tanh } e^{-2x} \tag{7}$$

In analogy to the procedure for the sine–Gordon equation the 2-soliton solution in its rest frame is obtained from the Ansatz (Hoenselaers and Micciche, 2001)

$$\varphi = 4\ Ar\ \mathrm{Tanh}\,(a(x)b(t))$$

a and b have to satisfy

$$\partial_x a^2 = c_o + cx^2 + c_4 x^4$$

$$\partial_t b^2 = c_4 + (c-1)\,x^2 + c_o x^4$$

The 2-soliton solution is given by either $c_0 = 0$ or $c_4 = 0$. A typical solution, for $c_0 = 0$, reads

$$\begin{aligned} a &= \sqrt{\frac{c}{c_4}}\frac{1}{\mathrm{Sinh}\,(\sqrt{c}x)} \\ b &= \sqrt{\frac{c_4}{c-1}}\mathrm{Sinh}\left(\sqrt{c-1}\quad t\right) \end{aligned} \tag{8}$$

Generally, the behaviour of Sinh–Gordon solitons is rather similar to that of sine–Gordon solitons with the exception that Sinh–Gordon solitons are given by singularities rather than wave crests. The Lagrangian is given by

$$L = \frac{1}{2}(\partial_x\varphi^2 - \partial_t\varphi^2) + \mathrm{Cosh}\,\varphi$$

and the Hamiltonian, the value of which, for a given solution, is the energy density, reads (the additive constant stems from the convention to have the lowest state at zero energy.)

$$H = \frac{1}{2}(\partial_x\varphi^2 + \partial_t\varphi^2) + \mathrm{Cosh}\,\varphi - 1 \tag{9}$$

For the 1-soliton solution the energy density is

$$E = 8\ e^{2x}\quad (1 - e^{2x})^{-2}$$

While the integral over this energy density does not exist, its Cauchy principal value does exist and

$$\oint_{-\infty}^{+\infty} E\ dx = 4$$

We now turn to solutions of the Cosh–Gordon equation. In this case the equation reads

$$\partial_x^2\varphi - \partial_t^2\varphi = \mathrm{Cosh}\,\varphi \tag{10}$$

and, looking for a travelling wave solution in its rest frame, the analogue of (7) becomes

$$\frac{1}{2}\partial_x\varphi^2 = \text{Sinh}\,\varphi - E$$

This right-hand side of this equation does not admit a double zero and thus there are no solutions in terms of elementary functions. A solution is given by

$$\varphi = -2\ln\left(\frac{1}{\sqrt{k}}cn\left(\frac{x}{2\sqrt{k}}, k\right)\right), \quad E = \frac{1}{2}\left(k + \frac{1}{k}\right) \tag{11}$$

The Hamiltonian for the Cosh–Gordon equation, viz.

$$H = \frac{1}{2}(\partial_x\varphi^2 + \partial_t\varphi^2) + \text{Sinh}\,\varphi \tag{12}$$

is not bounded from below and thus there is no ground state.

4. CONCLUDING REMARKS

We have shown that integrability of an equation is not necessarily related to the existence of solitons. This is, in particular, the case if the equation in question does not admit a trivial solution and the Hamiltonian does not have a ground state.

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